II.4. BOOK CIRCULATION INTERFERENCE

In this chapter we will study the book circulation process in a library. Our approach is based on Morse (1968; Chapter 4). Two special cases will be studied : the first one is the case in which a person gives up immediately when the desired book is not present. The second case is the one in which a library patron, finding that a book is on loan, always places a reservation. This second situation obviously gives rise to an application of queueing theory. Every book then stands for a separate queue. Service is a loan and the service time is the time between the moment the book is taken from the shelf and the moment it is put back in place by the library assistant. The queue itself consists of those persons who have placed a reservation because the book is on loan.

Somewhat surprisingly, the first situation can also be dealt with by using queueing theory. Indeed, it can be considered as a queueing system where there is no waiting room : when servers are busy (i.e. when the book or books are on loan) customers have to leave the system.

II.4.1. The general situation : some notation

When a library owns a particular item and this item is allowed to circulate, several situations may occur. The most important ones - amongst which are those we will study - are indicated in the flow chart of Fig.II.4.1.

The two cases we will study in detail can be considered as extreme cases. In this way we hope to obtain enough information about intermediate, more realistic situations.

A note on notation. We assume that λ persons a year wish to lend a particular item and that these λ arrivals are randomly distributed over the year. The number of circulations a year if a book were withdrawn by someone else immediately after its return is denoted as μ. Hence $1/\mu$ equals the average time the item is not on the shelf, so that $1/\mu$ is the mean loan period, including handling time.

The average number of times an item is effectively loaned out during a year is denoted by R. Then $R \leq \lambda$ and $R < \mu$. We note that in reality the mean loan period (service time) $1/\mu$ is not completely arbitrary because there is a maximum period an item is allowed to be withdrawn.

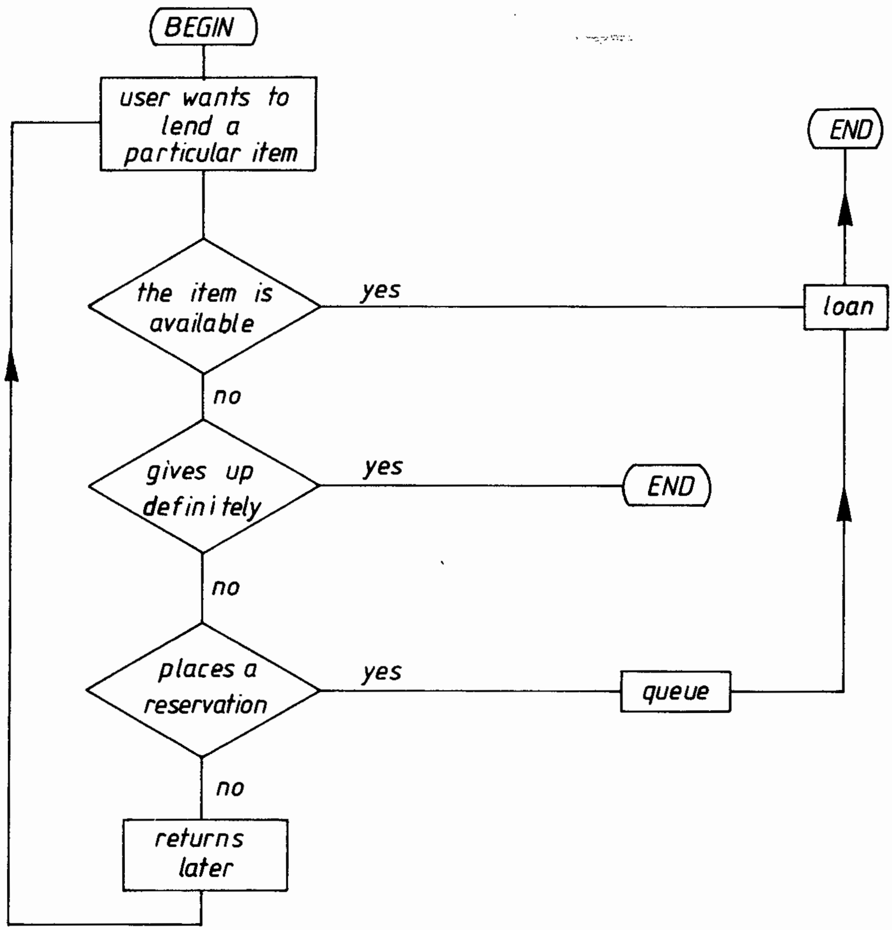

Fig.II.4.1 Flow chart describing demands for a particular item

II.4.2. First special case : complete balking

This case (cf. Fig.II.4.1) yields a queueing situation with *no waiting facilities*. The average fraction of the year an item is not on the shelf is given by the expression

$$R\ (1/\mu)\ ,$$

this is : the average number of loans times the average loan period. We note that $R < \mu$ implies that $R/\mu < 1$ (as it must be).

As there are on the average λ persons a year who want to lend the item, there are $\lambda(R/\mu)$ who do not find the item on the shelf and give up. Consequently, there are $\lambda - \lambda R/\mu$ who can and will lend it. This number is by definition equal to R. This reasoning yields :

$$R = \lambda - \lambda(R/\mu)$$

or

$$R = \frac{\lambda\mu}{\lambda + \mu} \qquad [II.4.1]$$

or

$$\lambda = \frac{R\mu}{\mu - R} \ . \qquad [II.4.2]$$

We use P_0 to denote the probability that an item will not be available and P_1 to denote that it is. Then

$$P_0 = R/\mu = \lambda/(\lambda+\mu) \qquad [II.4.3]$$

and

$$P_1 = 1 - P_0 = 1 - \lambda/(\lambda+\mu) = \mu/(\lambda+\mu) \ . \qquad [II.4.4]$$

The average number of persons who do not find the item, denoted by U, is then :

$$U = \lambda - R = \frac{\lambda R}{\mu} = \frac{\lambda^2\mu}{\mu(\lambda+\mu)} = \frac{\lambda^2}{\lambda + \mu} \ . \qquad [II.4.5]$$

We say that U is the degree of dissatisfaction; it is also the average non-satisfied demand. As a function of μ and R, U is given by :

$$U = \frac{R^2}{\mu - R} \ . \qquad [II.4.6]$$

Some examples.

Example 1 : $\mu = 15$ (i.e. an average loan period of 24 days).

Table II.4.1

λ	0	1	2	3	5	10	20	100
R	0	0.94	1.76	2.5	3.75	6	8.6	13
U	0	0.06	0.24	0.5	1.25	4	11.4	87

Example 2 : $\mu = 20$ (i.e. an average loan period of 18 days)

Table II.4.2

λ	0	1	2	3	5	10	20	100
R	0	0.95	1.82	2.61	4.0	6.67	10	16.67
U	0	0.05	0.18	0.39	1.0	3.33	10	83.33

Example 3 : $\mu = 24\frac{1}{3}$ (i.e. an average loan period of 15 days)

Table II.4.3

λ	0	1	2	3	5	10	20	100
R	0	0.96	1.85	2.67	4.15	7.09	10.98	19.57
U	0	0.04	0.15	0.33	0.85	2.81	9.02	80.43

Comparing Tables II.4.1, II.4.2 and II.4.3 shows the influence of shortening the loan period on the degree of dissatisfaction. For μ fixed, $U = R$ when $\mu = \lambda$.

II.4.3. Second case : every potential lender places a reservation when the item is not immediately available

This case leads to a *queueing situation* as studied in the preceding chapter. We assume an (M|M|1) queue for every item in the library. To imitate a realistic situation, the utilisation factor $\rho = \lambda/\mu$ must be smaller than one.

We take $\lambda = R$, since we may assume ($\rho < 1$!) that all arrivals are eventually able to lend the desired item. According to equation [II.3.5], the average number of persons on the reservation list is given by :

$$\bar{N}_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2/\mu^2}{1-\lambda/\mu} = \frac{R^2}{\mu(\mu-R)} \quad . \qquad [II.4.7]$$

The average time these persons have to wait for the reserved item is

$$\bar{T}_q = \frac{\bar{N}_q}{\lambda} = \frac{R}{\mu(\mu-R)}$$

(by means of [II.3.8]).

The probability that the desired item will not be available (i.e. the average fraction of the time that the item is in service) is then

$$P_0 = \frac{\lambda}{\mu} = \frac{R}{\mu} \ (= \rho) \ ;$$

and the probability that the item will be there is given by

$$P_1 = 1 - \frac{R}{\mu} = \frac{\mu - R}{\mu} \quad .$$

As a measure of dissatisfaction (U) we use here the average total number of people in the queue during one year; this is the average number of persons on the reservation list, multiplied by the number of loans (μ) :

$$U = \bar{N}_q \mu = \frac{R^2}{\mu - R} \quad . \qquad [II.4.8]$$

We get the same result as for the first special case [II.4.6], although the meaning of U is not the same. In either case, the degree of dissatisfaction in any situation is given by

$$U = \frac{R^2}{\mu - R} \quad , \qquad [II.4.9]$$

which could then be used in all intermediate cases.

Examples.

1) $1/\mu = 1/15$ (average loan period of 24 days)

Table II.4.4

R	0	1	2	3	5	10	14	
$\bar{N}_q$	0	0.005	0.02	0.05	0.17	1.33	13.07	
$\bar{T}_q$	0	1.7	3.7	6.1	12.2	48.7	341	in days
U	0	0.07	0.3	0.75	2.5	20.0	196	

2) $1/\mu = 1/20$ (average loan period of 18 days)

Table II.4.5

R	0	1	2	3	5	10	19	
$\bar{N}_q$	0	0.003	0.01	0.03	0.08	0.5	18.1	
$\bar{T}_q$	0	0.96	2.03	3.2	6.1	18.3	345	in days
U	0	0.05	0.2	0.5	1.7	10	361	

3) $1/\mu = 1/26$ (average loan period of 14 days)

Table II.4.6

R	0	1	2	3	5	10	20
$\bar{N}_q$	0	0.002	0.006	0.015	0.046	0.24	2.56
$\bar{T}_q$	0	0.56	1.2	1.8	3.3	8.8	46.8 in days
U	0	0.04	0.17	0.4	1.2	4.8	66.7

If the degree of dissatisfaction is too high, measures have to be taken : the loan period must either be shortened (in which case the above models can still be used) or additional copies must be bought. However, we will show in the next section that, in the case of complete balking, it is better to buy two copies than to halve the loan period (and have only one copy available).

II.4.4. Multiple copies

II.4.4.1. Complete balking

The probability that a user wishes to lend a particular book and finds no copy on the shelves is denoted by P_0. When there are m copies of this book, the situation can be described by an (M|M|m) queue with no waiting capacity, i.e. a queueing system where it is impossible to wait to be served (we need this rather special situation, because we assume complete balking). It can be shown (see Phillips et al. (1976; p.305)), that

$$P_0 = \frac{\frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m}{\sum_{i=0}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i} \quad , \qquad [II.4.10]$$

an equation known in the literature as Erlang's lost call equation.
In particular for m = 2 we find :

$$P_0 = \frac{\lambda^2}{2(\mu^2 + \mu\lambda + \frac{\lambda^2}{\mu})} \qquad [II.4.11]$$

and for m = 3 :

$$P_0 = \frac{\lambda^3}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3} \quad . \qquad [II.4.12]$$

Note that for m = 1 we have $P_0 = \frac{\lambda/\mu}{1 + \lambda/\mu} = \frac{\lambda}{\lambda + \mu}$, which is exactly the same formula found in Section II.4.2 by a more elementary reasoning. Further we have $U = \lambda P_0$ and $R = \lambda - U$ (where R denotes the mean circulation per year of all m copies).

Examples : $\mu = 24\frac{1}{3}$

Table II.4.7

	$\lambda = 10$	$\lambda = 20$
R 1 copy	7.09	10.98
2 copies	9.44	16.87
3 copies	9.92	19.18

We will now compare the cases in which m = 2, μ, λ, and m = 1, μ/2, λ. For m = 2, $R = \lambda - \lambda \frac{\lambda^2}{2(\mu^2 + \mu\lambda + \lambda^2/2)}$. For m = 1 and μ/2 we have by means of [II.4.1] : $R' = \frac{\lambda\mu/2}{\lambda + \mu/2}$. This yields R > R'. Indeed :

$$R > R'$$

$$\Longleftrightarrow$$

$$\lambda - \lambda \frac{\lambda^2}{2(\mu^2 + \mu\lambda + \lambda^2/2)} > \frac{\lambda\mu/2}{\lambda + \mu/2}$$

$$\Longleftrightarrow$$

$$\lambda^2\mu(4\mu + 3\lambda) > 0 \quad ,$$

which is always satisfied. Hence, if one can spend the money, it is always better to buy a second copy (and leave the lending period as it is) than to continue with one copy and halving the lending period.

II.4.4.2. The reservation model

Here we have an (M|M|m) queue with an infinite waiting capacity, i.e. the queueing system studied in Section II.3.3. Hence

$$\rho = \frac{\lambda}{m\mu} \; ; \quad \bar{N}_s = \rho m \; ; \quad \bar{N}_q = \frac{(m\rho)^{m+1}}{K\ m\ m!\ (1-\rho)^2}$$

with

$$K = \left(\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}\right) + \frac{(m\rho)^m}{m!\,(1-\rho)} \quad ;$$

$$\bar{T}_q = \frac{1}{\lambda} N_q \quad \text{and} \quad \bar{T}_s = \frac{1}{\mu} \quad .$$

$$P_0 = \rho = \frac{\lambda}{m\mu} \quad .$$

Some examples : μ = 24 1/3

Table II.4.8

		λ = 10	λ = 20
R	1 copy	5.89	3.56
	2 copies	7.95	11.78
	3 copies	8.63	14.52

II.4.5. Notes and comments

To be able to apply the equations in the preceding section, we have to know μ, λ and R. The parameter $1/\mu$ can easily be determined from known data (and is influenced by rules on the circulation time). The parameter λ is more difficult to assess. In the reservation model λ = R, in the case of complete balking, $\lambda = R\mu/(\mu-R)$ (case of one copy). If R is known, λ is also known. While the average loan period is rather stable, λ and R are not. We can only approximate λ by using the values for R of the last year(s). This problem will be studied in the next chapter.

A more detailed approach to library circulation interference can be found in Morse (1968; Chapter 4) and in Morse (1972, 1976, 1979). In these follow-up papers Morse estimates the potential demand for library material and uses these estimates to study the effects of a change in the allowed length of a loan period, and what the result on the average per-book circulation would be if duplicate copies were bought for all books that circulated more than a fixed number of times.

II.5. MARKOV PROCESSES AND MORSE'S MODEL

II.5.1. Stochastic processes - Markov processes

II.5.1.1. Stochastic processes

A *stochastic process* is any system of functions governed by probabilistic laws. When time is measured discretely, we have a discrete time stochastic process denoted by (S_n; n = 1,2,3,...). The distinct values the process can assume are called 'states', and the set of all states is called the 'state space'. The variable S_n denotes the state at time n. Changes of state are referred to as 'transitions'. When a discrete stochastic process can assume only a finite number of states, the structure of allowable transitions can be pictured in some kind of digraph (cf. Section II.2.1), termed a 'transition diagram'. States are represented by nodes and transitions by arrows. The stochastic process can be thought of as the random walk of a particle over the transition diagram (Fig.II.5.1).

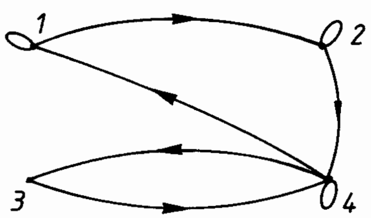

Fig.II.5.1 A transition diagram

II.5.1.2. Markov processes

A *Markov process* is a simple form of a stochastic process having the property that the conditional probability of an outcome only depends on its immediately preceding outcome, and not on any of the previous outcomes.

A discrete or finite Markov chain is then a stochastic process that satisfies the following three requirements :

1) The process must be a discrete-time process, that is, the movement of the particle among the states must occur at finite intervals.
2) The process must have a finite set of states.
3) The process must possess the Markov property, meaning that the probability of each outcome relies solely on the outcome of its immediate predecessor.

As in this case the state S_n depends only on S_{n-1}, it is sufficient to know the probabilities of going from one state to another. These one-step transition probabilities are recorded in a transition matrix P, where p_{ij} denotes the probability that a particle which is in state i at time n will be in state j at time n+1. As the p_{ij}'s are probabilities, we have for every i : $\sum_j p_{ij} = 1$. Formally speaking, p_{ij} is the conditional probability $P(S_{n+1} = j|S_n = i)$.

Instead of using a matrix, we could make the transition diagram (a digraph) into a weighted digraph, where weights are transition probabilities. See Fig.II.5.2 (note that p_{34} must be 1).

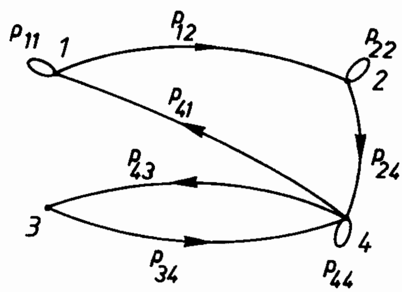

Fig.II.5.2 A labelled (weighted) transition diagram

We will now assume that these transition probabilities will not change in time. This is the reason why we used the notation p_{ij} (independent of n). The Markov process is then called a 'stationary Markov process'. For such Markov processes the transition matrix $P = (p_{ij})$ is all that is needed to describe the entire process.

Indeed, the n-step transition probabilities $p_{ij}^{(n)}$ are defined by $p_{ij}^{(n)} = P(S_n = j|S_0 = i)$. In other words, $p_{ij}^{(n)}$ is the probability that the process will be in state j at time n, given the fact that it was in state i at time 0. One can now show that the matrix $P^{(n)} = (p_{ij}^{(n)})$ is obtained from the matrix multiplication $P^{(n-1)}.P$ and hence by induction $P^{(n)} = P^n$. That is, the matrix of n-step probabilities is merely the transition matrix raised to the n^{th} power (i.e. multiplied by itself n times).

What happens when n, the number of transitions, goes to infinity? This limiting distribution is termed the 'steady state' or the 'stationary' distribution. In most cases (there are exceptions) the initial state will become less and less relevant to the n-step transition probability as n increases. Then, denoting $p_j^{(n)} = P(S_n=j)$,

$$\pi_j = \lim_{n\to\infty} p_j^{(n)} = \lim_{n\to\infty} P(S_n = j) = \lim_{n\to\infty} P(S_n = j|S_0 = i) = \lim_{n\to\infty} p_{ij}^{(n)}$$

Whenever it is true that the steady state probabilities are independent of the initial state, the matrix $P^{(n)}$ becomes, as n goes to infinity, a matrix in which the rows are identical. Each row becomes, in fact, the vector $\pi = (\pi_1,\pi_2,\ldots)$, where $\Sigma\pi_k = 1$. Table II.5.1 illustrates this phenomenon.

Table II.5.1. A Markov process

j	0	1	2	3	4	5	6	7
				P				
i								
0	.819	.164	.016	.001	-			
1	.606	.303	.076	.013	.002	-		
2	.449	.300	.144	.038	.008	.001	-	
3	.333	.366	.201	.074	.020	.005	.001	
4	.247	.345	.242	.113	.039	.011	.002	.001
5	.183	.310	.264	.149	.064	.022	.006	.002
6	.135	.271	.271	.180	.090	.036	.012	.005
				P^2				
0	.777	.190	.028	.004	.001	-		
1	.719	.224	.047	.009	.001	-		
2	.665	.251	.065	.015	.003	.001	-	
3	.616	.274	.083	.021	.005	.001	-	
4	.570	.291	.101	.029	.007	.002	-	
5	.527	.304	.118	.037	.010	.003	.001	-
6	.487	.314	.134	.046	.014	.004	.001	-
				P^4				
0	.762	.199	.033	.005	.001	-		
1	.758	.201	.035	.005	.001	-		
2	.752	.204	.037	.006	.001	-		
3	.747	.207	.038	.007	.001	-		
4	.742	.210	.040	.007	.001	-		
5	.527	.304	.118	.037	.010	.003	.001	-
6	.487	.314	.134	.045	.014	.004	.001	-
				steady state				
all i	.761	.199	.034	.005	.001	-		

The steady state probabilities have several useful interpretations. If you fix a point in time in the distant future, π_j is the probability that you will find the process in state j at that time. It can also be viewed as a time average : if you ran the process for a long time, π_j would be the fraction of time that the process spent in state j. Finally, it can be viewed as the reciprocal of the mean number of transitions between recurrences of the state. For example, if $\pi_4 = 0.2$, an average of 5 transitions will occur before the system will be back in state 4.

II.5.2. Morse's Markov model for book use

II.5.2.1. Determination of the Markov process

Let X_t denote the stochastic variable of the number of loans of a fixed group of books that already belong for t years to the library's collection. The distribution function and the characteristics of this stochastic variable depend on the type of collection being studied. The average number of loans in a given period of time (e.g. one year) will be denoted by R_t.

The stochastic variable of the number of loans of those books which were borrowed exactly m times ($m \in \mathbb{N}$) during the previous year is denoted by T_m. Then T_{mn} denotes the probability that a book that was on loan m times last year will be on loan n times this year. Finally, N(m) denotes the average number of loans this year of books that were on loan m times last year.

In general, one expects T_m, T_{mn} and N(m) to be dependent on time. However, Morse's model assumes a stationary process, implying that these quantities are independent of time.

Mathematically, we have the following relations :

$$E(X_t) = R_t \quad ; \qquad \text{[II.5.1]}$$

$$T_{mn} = P(T_m = n) = P(X_t = n \mid X_{t-1} = m) \quad ; \qquad \text{[II.5.2]}$$

$$E(T_m) = N(m) = \sum_{n=0}^{\infty} nT_{mn} \ . \qquad \text{[II.5.3]}$$

For every m :

$$\sum_{n=0}^{\infty} T_{mn} = 1 \ . \qquad \text{[II.5.4]}$$

Morse's model further assumes that loans follow a Markov process, where the T_{mn}'s are the entries of the corresponding transition matrix. Further, Morse assumes a fixed linear relation between N(m) and m : N(m) = a + bm (cf. Subsection I.3.8.4 and Table I.3.9). Note that the slope parameter b will lie between 0 and 1 as on the average the popularity of books tends to diminish. This slope parameter b can then be viewed as an indicator of the rate in which books of a particular population lose popularity : the smaller the value of b is, the greater the decline in popularity is.

Finally, Morse assumes T_m to be a Poisson distribution. Consequently, with the assumed form of N(m) :

$$T_{mn} = \frac{(a + bm)^n}{n!} e^{-(a + bm)} \quad . \qquad [II.5.5]$$

As this determines the transition matrix T, the whole process is firmly determined. The Markov matrices of Table II.5.1 are in fact those associated with Morse's model for b = 0.3 and a = 0.2. Whether or not this model applies to a particular situation has to be investigated in every case. However, Morse (1968) reports that his model frequently fits data.

II.5.2.2. <u>An application</u>

We consider a group of books where N(m) = a + bm. If $R_{t-1} = \bar{x}$, what will R_t be? Let K be the total number of books under consideration and let k_i be the (unknown) number of books that were loaned out i times in the period t-1. Then

$$\bar{x} = \frac{\sum_{i=0}^{\infty} ik_i}{K}$$

with $K = \sum_{i=0}^{\infty} k_i$.

Following Morse's model N(i) = a + bi, for every i. The number $k_iN(i)$ denotes the total number of loans (in period t) of those books that were loaned out i times in period t-1. Summing over all i :

$$\sum_{i=0}^{\infty} k_iN(i)$$

yields the total number of loans in period t. The average number of loans in period t is then :

$$\frac{\sum_{i=0}^{\infty} k_iN(i)}{K} = \frac{\sum_{i=0}^{\infty} k_i(a + bi)}{K} = a + b\bar{x} \quad .$$

This shows that $R_t = a + bR_{t-1}$ or $R_t - bR_{t-1} = a$. This is a difference equation for which the solution is :

$$R_t = \frac{a}{1-b} + (R_1 + \frac{a}{1-b})\, b^{t-1} \quad . \qquad [II.5.6]$$

As $0 < b < 1$, $\lim_{t\to\infty} R_t = \frac{a}{1-b}$, which is the limiting lending rate. Its value is a measure of the long-term popularity of this group of books.

By using [II.4.2], we see that in the case of complete balking the demand λ_t is also a function of time given by :

$$\lambda_t = \frac{R_t\mu}{\mu - R_t} \quad ,$$

where R_t is determined by [II.5.6] and $R_1 = \frac{\lambda_1\mu}{\lambda_1 + \mu}$.

Some examples.

We will consider the case in which $\mu = 25$, $R_1 = 12$ and different values of a and b; U_i denotes the degree of dissatisfaction (unsatisfied demand) in the case of i copies.

1) a = 0.05, b = 0.90

t	1	2	3
R_t	12	10.85	9.815
λ_t	23.08	19.17	16.16
U_1	11.08	8.32	6.34
U_2	4.19	2.73	1.82
U_3	1.84	1.11	0.38

2) a = 0.30, b = 0.60

t	1	2	3
R_t	12	7.5	4.8
λ_t	23.08	10.71	5.94
U_1	11.08	3.21	1.14
U_2	4.19	0.65	0.13
U_3	1.84	0.20	0.01

c) a = 0.45, b = 0.20

t	1	2	3
R_t	12	2.85	1.02
λ_t	23.08	3.22	1.06
U_1	11.08	0.37	0.04
U_2	4.19	0.02	0.00
U_3	1.84	0.00	0.00

These examples show, among other things, that if loans can be described by a Markov process, buying duplicate copies is seldom a wise decision, especially for those items that decline rapidly in popularity, e.g. a record library, where rock music usually shows a very rapid decline in popularity. Moreover, it is generally true that there is always a degradation in the effectiveness of extra copies, in the sense that when two copies are available, the number of loans does not double. These remarks are, however, not in contradiction with the results of Subsection II.4.4.1.

Table II.5.1 is actually an illustration of Morse's Markov model in the case in which b = 0.3 and a = 0.2. The steady state distribution shows, for example, that if such a book is loaned out twice a year, it will take - on the average - 1/0.034 ≈ 29 years before this will happen again.

II.5.3. Notes and comments

Hendricks (1972) describes arrangements of N books on a shelf. If the probability of selecting each book is known, if books are returned by placing them at one end of the shelf and if only one book is removed at a time, then the N! arrangements of the books are considered as states of a Markov chain and the stationary distribution is described. Burville and Kingman (1973) continued this work and found the probability that a given book will be in a given position on the shelf, in terms of the frequencies with which different books are demanded.

The relation $P^{(n)} = P^{(n-1)}P$ we have observed in Subsection II.5.1.2, is a special instance of a result in Markov theory, generally known as the Chapman-Kolmogorov equation, which states that for all $m,n \in \mathbb{N}_0$, $m < n$: $P^{(n)} = P^{(n-m)}P^{(m)}$.

Besides the discrete time stochastic and Markov processes, continuous time stochastic and Markov processes are also studied. Unfortunately, they have been infrequently applied in informetric studies. Movements of authors among subareas in a scientific discipline have been studied - using Markov chains - in Goffman (1971), Goffman and Warren (1980), Pao and McCreery (1986). The results on queueing theory presented in Chapter II.3 can be proved by Markov process methods.

Morse (1968) checked his model at MIT's Science Library and found quite satisfactory agreement between data and model. In general, he observed that the parameter $a \approx 0.4$ and $b \approx 0.5$ for books of five years of age or less and $a \approx 0.2$ and $b \approx 0.5$ for older books. From his analysis he predicted future use and gave suggestions on when to retire books and how to cope with lost books (and what this loss means to potential users). His book (Morse (1968)) also contains several tables of Markov-Poisson processes.

Chen (1976) pursued a study similar to Morse's in Harvard's Countway Medical Library. She found that much higher values for the parameter a and slightly lower values for b were appropriate for this library. Further, she confirmed the observation that the parameter a does decrease for older items. Chen also extended Morse's model by developing a procedure to deal with biased circulation data. More recently, the model was utilised to evaluate use patterns in an academic library and a resource library in the UK (Hindle (1979)) and to study the use of life sciences books at MIT (Spurlock and Yen (1978)).

An extensive study to test Morse's model was carried out by Beheshti and Tague (1984). Utilising eleven years of circulation transactions at the University of Saskatchewan, they showed that Morse's model fits approximately 99 % of the data, with values for a ranging from 0.37 to 0.43 and for b ranging from 0.31 to 0.34. Nevertheless, they also found that use of the Markov model creates some problems. First, the model does not fit the long tail of the distributions of the number of transactions per document. Once monographs circulate more than eight times per academic year, the correlation coefficient between the number of times the documents circulate in one year and the average number of times they circulate the following year decreases sharply.

Second, the value of a is time dependent, whereas b fluctuates more randomly with time. Third, they question the validity of applying the Poisson distribution to explain the actual transaction distribution about the mean N(m). Coady (1983) insists that direct tests of the Markovity of library circulation data should be conducted before models based on the Markov property are used. The relation of Morse's model to other models and more detailed criticisms of it will be discussed in the next chapter.

II.6. OTHER LIBRARY CIRCULATION MODELS

II.6.1. Burrell's simple stochastic model for library loans

With the concept of a self-renewing or no-growth library (Trueswell (1976)) in mind, Burrell (1980), see also Burrell and Cane (1982), presents a simple mathematical model for library loans. This model should help practising librarians in collecting and interpreting data, weeding little used books and purchasing extra copies, where necessary. The main objectives of such a simple model are :

(i) to contain only a small number of parameters, of which the meaning is easy to understand and which can be used to characterise the library;

(ii) to be described by parameters that are not too difficult to estimate;

(iii) to provide a qualitatively good fit for data from various libraries and for various time periods.

Later, Burrell (1984) explicitly referred to Sandison (1977) to defend his simple stochastic model. Indeed, in the paper referred to above, Sandison asserted that mathematical models proposed to assist librarians should : i) be based on valid assumptions, ii) be explained in sufficiently simple terms for the ordinary librarian to carry out and iii) result in better advice than that obtainable by simpler techniques. This is exactly what Burrell's simple model aims to do.

II.6.1.1. The straight-line phenomenon for frequency of circulation

As has often been observed, when plotting on semilogarithmic scales (with the y-axis as the logarithmic axis), the number of times (r) an item has circulated in a fixed period T versus the number of items ($f_r(T)$), the resulting curves take one of three shapes : see Fig.II.6.1 (cf. Montgomery et al. (1976), Burrell (1980, 1982), Burrell and Cane (1982), Leemans et al. (1989)).

In this section we will study the simple case (a), where we observe the so-called *straight-line phenomenon*. The other cases (b) will be dealt with in Section II.6.2.

The straight-line phenomenon means that : $\log (f(r)) = a + br$, with $b < 0$ (as the line is decreasing). This yields further :

$$f(r) = e^{a+br}$$

or

$$f(r) = C\delta^{r}$$

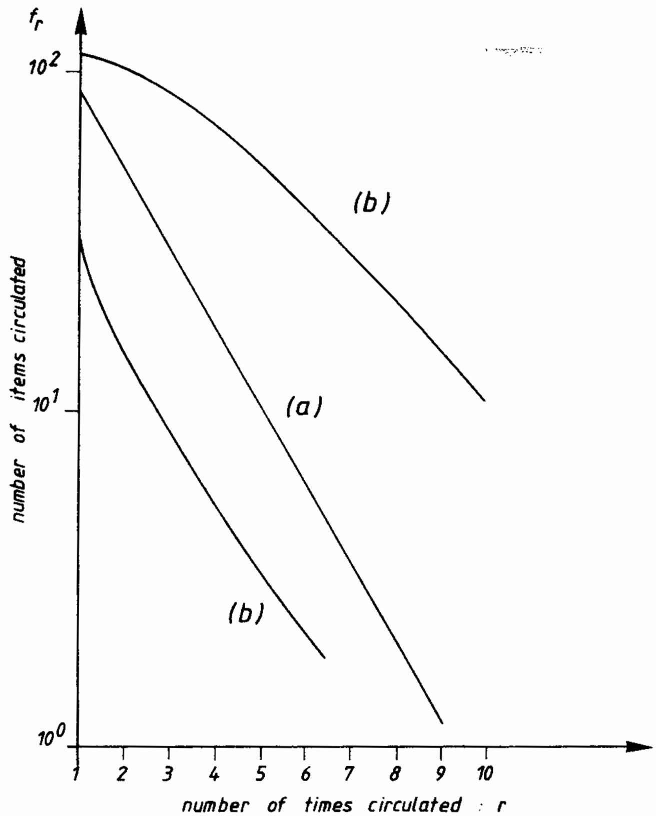

Fig.II.6.1 Three basic shapes of circulation distributions (semilogarithmic scales)

with

$$C = e^a \quad \text{and} \quad \delta = e^b \ . \qquad [II.6.1]$$

If the size of the total collection in the period T is N, the probability that a book will be on loan exactly r times, denoted $P_r(t)$, is given by

$$P_r(t) = \frac{C}{n}\,\delta^r \quad \text{with} \quad \sum_{r=0}^{\infty} \frac{C}{N}\,\delta^r = 1 \ . \qquad [II.6.2]$$

Hence $\frac{C}{N}\frac{1}{1-\delta} = 1$ or $C = N(1-\delta)$. So, the probability that a book will be on loan exactly r times is $(1-\delta)\delta^r$, meaning that we have a geometric distribution. Absolute frequencies $f_r(T)$ are then given by $N(1-\delta)\delta^r$. The average number of loans is then $\mu = \delta/(1-\delta)$ with a variance of $\delta/(1-\delta)^2$ (see Subsection I.2.4.4).

Burrell assumes μ to be proportional to T. This yields :

$$\mu_T = \frac{\delta_T}{T - \delta_T} = \frac{1}{\alpha} T \quad , \qquad [II.6.3]$$

where $1/\alpha$ is a proportionality factor. Solving [II.6.3] for δ_T results in :

$$\delta_T = \frac{T}{\alpha + T} \quad . \qquad [II.6.4]$$

As $\mu = \frac{1}{\alpha} T$, we can interpret α as the average time between two successive loans. Clearly, α will depend on such factors as the size of the collection, the potential reader population, loan policies and so on.

II.6.1.2. A model to explain the straight-line phenomenon

Every book in a library has a certain desirability, depending on its popularity or usefulness. As a definition for this desirability, Burrell takes the average number of times this book is on loan during a unit period of time. A year is usually a convenient unit.

For a book with a desirability of λ, it is assumed that actual loans occur following a Poisson distribution. This means that the probability that this book will be loaned out k times during a period of length T is given by

$$\frac{e^{-\lambda T} (\lambda T)^k}{k!} , \qquad k = 0,1,2,\ldots \quad . \qquad [II.6.5]$$

Finally, we also need an assumption about the distribution of the desirability over all documents. Burrell uses a negative exponential distribution with a parameter of $1/\alpha$. This is a continuous distribution with a density function of $f(\lambda) = \alpha e^{-\alpha\lambda}$, $\lambda \geq 0$ (see Subsection I.2.4.5).

$P_r(T)$, the probability that a book will be loaned out r times during a period T is then :

$$P_r(T) = \int_0^\infty \frac{e^{-\lambda T} (\lambda T)^r}{r!} \alpha e^{-\alpha\lambda} \, d\lambda$$

$$= \frac{\alpha}{r!T} \int_0^\infty e^{-\lambda T(1 + \frac{\alpha}{T})} (\lambda T)^r \, d(\lambda T)$$

$$= \frac{1}{(1 + \frac{\alpha}{T})^{r+1}} \frac{\alpha}{r!T} \int_0^\infty e^{-(\lambda T(1 + \frac{\alpha}{T}))} (\lambda T(1 + \tfrac{\alpha}{T}))^{(r+1)-1} \, d(\lambda T(1 + \tfrac{\alpha}{T}))$$

$$= \frac{\alpha r!}{(1 + \frac{\alpha}{T})^{r+1} \, r!T} \qquad \text{(by the definition of the } \Gamma\text{-function)}$$

$$= \frac{\alpha}{T(1+\frac{\alpha}{T})^{r+1}}$$

$$= (1 - \frac{T}{T+\alpha})(\frac{T}{T+\alpha})^r \quad .$$

This is a geometric distribution (Subsection I.2.4.4) with a parameter of $\delta = \frac{T}{T+\alpha}$, explaining, in a way, Subsection II.6.1.1, formula [II.6.4].

In conclusion, when we assume the number of loans to be distributed according to a Poisson distribution with a parameter of λT (where λ is the desirability per unit of time) and when we further assume that desirabilities themselves are distributed according to a negative exponential distribution, then loan frequencies over a period T are distributed according to a geometric distribution with a mean of T/α, where α depends on the specific collection.

II.6.1.3. The zero class : dead items

Burrell assumes that there is a fraction $(1-\beta)$, $0 \leq \beta \leq 1$, of documents in the collection that never circulate : books on permanent reserve, lost or stolen books, redundant books (e.g. superseded by more recent, corrected editions). These items merely serve to inflate the *zero class*. Note also that they are distinct from items which, though candidates for circulation, happen not to have circulated in the observed period. Although part of these 'dead' items are well known to be dead (e.g. those on permanent reserve), others are not. Their fraction $(1-\beta)$ must be estimated from observed circulation data. The analysis of Subsections II.6.1.1 and II.6.1.2 then only refers to a fraction β of the collection. In that case, two parameters α and β (or β and δ, because α and δ are related) must be estimated.

For a period T, so-called maximum likelihood estimators (a kind of 'best' estimator) for β and δ (denoted by $\hat{\beta}$ and $\hat{\delta}$) are :

$$\hat{\delta} = 1 - \frac{N - f_0(T)}{\sum_{r=0}^{\infty} rf_r(T)} \quad \text{and} \quad \hat{\beta} = \frac{N - f_0(T)}{N\,\hat{\delta}} \quad \text{(Burrell (1980; p.131))} \; .$$

Here $N - f_0(T)$ is the number of documents loaned out at least once and $\sum_r rf_r(T)$ is the total number of loans. These two numbers are easily available.

Then, by using [II.6.4],

$$\hat{\alpha} = \frac{T(1-\hat{\delta})}{\hat{\delta}} \; .$$

Fittings can be done recursively. If $E_r(T)$ denotes the expected number of books that are loaned out r times during period T (according to this model), then, as is clear from the previous sections,

$$E_0 = N((1-\hat{\beta}) + \hat{\beta}(1-\hat{\delta})) = N(1-\hat{\beta}\hat{\delta}) = N(1 - \frac{N - f_0(T)}{N}) = f_0(T) \quad [II.6.6]$$

$$E_r(T) = N\hat{\beta}(1-\hat{\delta})\hat{\delta}^r, \; r > 0 \; ; \quad [II.6.7]$$

hence

$$E_1(T) = N\hat{\beta}(1-\hat{\delta})\hat{\delta} = (N - f_0(T))(1-\hat{\delta}) \quad [II.6.8]$$

and

$$E_{r+1}(T) = \hat{\delta}E_r(T), \; r > 1 \quad [II.6.9]$$

by means of [II.6.7].
Examples can be found in Burrell (1980) where this model is fitted to data obtained from the Sussex University Library, the Wishart Library in Cambridge (UK) and the Pittsburgh University Library.

II.6.1.4. Longitudinal studies and applications to stock relegation

The probability that an item will not circulate during a period T is $(1-\beta) + \beta(1-\delta)$, where δ depends on T and equals $T/(\alpha+T)$. Hence this probability is :

$$1 - \beta\delta = 1 - \frac{\beta T}{\alpha + T} . \quad [II.6.10]$$

The fraction of the collection that circulates at least once during period T is then $\beta\delta = \beta T/(\alpha+T)$. The fraction of the loanable items that are loaned out at least once in period T, is thus $\frac{T}{\alpha+T}$.

For the Sussex Library, $\beta = 0.3788$ and $\alpha = 1.009$, so that taking $T = 5$ years yields $\beta T/(\alpha+T) = 0.315$. For this library the dead part is $(1-\beta) = 0.6212$. This means that about $100 \frac{T}{\alpha+T}$ % = 83 % of the loanable items in this library are loaned out at least once in 5 years.

Next, let us find the probability that a book will be loaned for the first time during the n^{th} period (here a period is taken equal to one year). The probability that a document will be on loan for the first time in the period between t_1 and t_2 ($t_1 < t_2$) is denoted by $\theta(t_1,t_2)$.

Let S_2 be the set of books that are on loan at least once in the period

$[0,t_2]$ and S_1 the set of books that is on loan at least once in the period $[0,t_1]$. We are looking for $P(S_2 \setminus S_1)$: the probability that an item will belong to the difference between these two sets. By using [II.6.10], this becomes :

$$\frac{\beta t_2}{\alpha + t_2} - \frac{\beta t_1}{\alpha + t_1} = \beta \left(\frac{t_2}{\alpha + t_2} - \frac{t_1}{\alpha + t_1}\right) . \qquad [II.6.11]$$

If stock is relegated on the basis of usage, one will ideally wish to relegate those items which have a low desirability. As the desirability of a particular item is unknown, a relegation policy will depend on past circulation figures. Burell studied the effect of adopting such as policy to relegate all items which have not been loaned out by time T, except those on permanent reserve.

The effect of this policy is that the retained and relegated collections have different desirability distributions. For the relegated stock we find a geometric distribution with added zeros, but for the retained stock the loan distribution arises as a difference between two geometric series. The following results can then be shown (see Burrell (1980)) :

proportion of collection which is relegated or on permanent reserve : $1 - \beta T/(\alpha + T)$ (via [II.6.10]);

mean usage (per annum) of retained stock : $\frac{1}{\alpha}\left(1 + \frac{\alpha}{\alpha + T}\right)$;

mean usage (per annum) of relegated stock (excluding dead items) : $1/(\alpha + T)$;

proportion of loans which require items from relegated stock : $\left(\frac{\alpha}{\alpha + T}\right)^2$.

We see that this relegation policy has the effect of increasing the mean usage of the retained collection by a factor of $(1 + \alpha/(\alpha + T))$ (bear in 'mind' that the average usage per year for the entire collection is $1/\alpha$).

When we take T = 9, this policy implies, for the case in which $\alpha = 1$ and $\beta = 0.7$, that we relegate 37 % of the collection, the mean usage of the retained stock will then be 1.1 (i.e. an increase of 10 %), while only 1 % of the requests will require use of the relegated stock (apply the above equations).

II.6.2. More refined models

II.6.2.1. Gamma mixture of Poisson processes with added zeros

Burrell's simple model starts from the so-called straight-line phenomenon. However, this may sometimes be too simplistic, and more or less greater deviations from this straight line are observed in real data. A generalisation is obtained by assuming that observed loan frequencies over a period T, $f_r(T)$, r = 0,1,2,..., fit a negative binomial distribution (which generalises the geometric distribution). Loans for a particular book might occur at random (i.e. following a Poisson distribution) at rate λ, while the distribution of λ over the population of books is described by a gamma distribution, generalising the negative exponential one has used in the simple case. We will show that these assumptions lead to an observed frequency distribution which is negatively binomial. There is also a dead class to be accounted for, yielding 'added zeros' (Burrell and Cane (1982), Burrell (1982)). We will assume that a fraction $1 - \beta$ belongs to this dead class; hence there is a fraction β of active items.

The gamma distribution is a very useful and flexible theoretical probability distribution defined on the positive real numbers. A particular gamma distribution is identified by the values of a parameter $\xi > 0$ and an index $\nu > 0$. The parameter ξ is associated with a scale of measurement (e.g. the time scale used when counting numbers of circulation), while the index ν tells us something about the underlying shape of the distribution (see Fig. II.6.2).

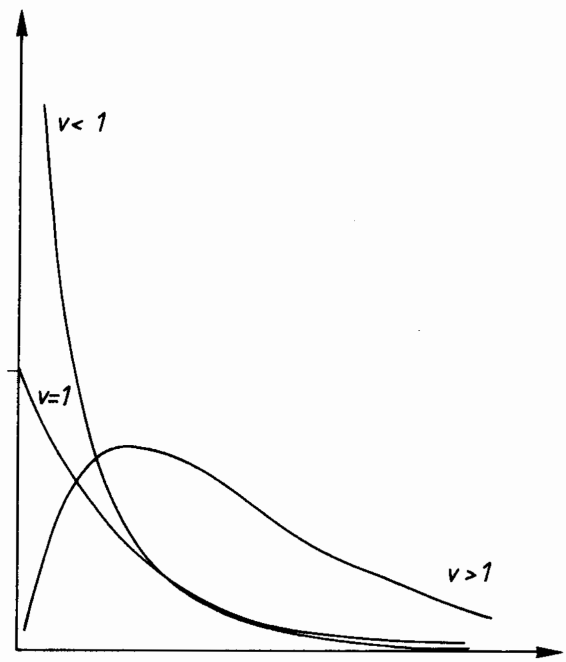

Fig.II.6.2 Gamma density function : basic shapes

The gamma density function is then given as :

$$g(\lambda) = \frac{\xi^{-\nu}\,\lambda^{\nu-1}\exp(-\lambda/\xi)}{\Gamma(\nu)}, \quad \lambda \geq 0, \qquad \text{[II.6.12]}$$

where in a library context λ corresponds to the mean lending rate of a typical active item. The mean of $g(\lambda)$ is $\nu\xi$ and its variance is $\nu\xi^2$; $\Gamma(\nu)$ denotes the gamma function (cf. [I.2.24]) from which the distribution gets its name. When $\nu = 1$, the gamma distribution becomes a negative exponential distribution, with a parameter of ξ (see Subsection I.2.4.5). When $\xi = 2$ and $\nu = n/2$, we obtain the χ^2-distribution (Subsection I.2.4.6) with n degrees of freedom.

Hence, if $P_r(T)$ denotes the probability that an arbitrarily chosen book will be loaned out r times in a period of length T, then, for $r > 0$:

$$\begin{aligned} P_r(T) &= \int_0^\infty \beta\,\frac{(\lambda T)^r e^{-\lambda T}}{r!}\,g(\lambda)\,d\lambda \\ &= \frac{\beta\Gamma(r+\nu)}{r!\,\Gamma(\nu)}\,p(T)^\nu\,q(T)^r \\ &= \beta\,\frac{(r+\nu-1)\ldots(\nu)\Gamma(\nu)}{r!\,\Gamma(\nu)}\,p(T)^\nu\,q(T)^r \\ &= \beta\binom{r+\nu-1}{r}\,p(T)^\nu\,q(T)^r \end{aligned}$$

and $P_0(T) = (1-\beta) + \beta p(T)^\nu$, where $p(T) = (1+\xi T)^{-1} = 1 - q(T)$.
This is a negative binomial distribution with parameters ν and $p(T)$ and with added zeros (Burrell 1982)). The whole model is usually described as a gamma mixture of Poisson processes.

This model is well known, having been originally investigated by Greenwood and Yule (1920) in connection with accident data. Since then it has been applied, for instance, by Wise (1946) in an industrial sampling context, by Arbous and Sichel (1954) in connection with absenteeism data, by Spilerman (1970) to describe racial disturbances in the USA, and by Morse (1976) in connection with library circulation statistics.

We further note that for active books the mean circulation is $\nu q(T)/p(T) = \nu\xi T$ (Subsection I.2.4.4), where $\nu\xi$ is the mean lending rate of the active population.

Wall (1980) and Ravichandra Rao (1982) suggested using the more general negative binomial distribution instead of the geometric one. Although the mixed

Poisson distribution gives better fits to the data in the Pittsburgh and the Wishart libraries - see Bagust (1983), who nevertheless does not believe in (and therefore does not use) a dead category - it involves the estimation of three parameters : one, denoted β, to determine the fraction of active items and two, $p(T)$ and ν, for the negative binomial distribution. The negative binomial distribution, without a dead category, was fitted to circulation data in the Huddersfield Public Library by Bagust (1983). This prompted Brownsey and Burrell (1986) to study the frequency of the circulation distribution of 16 public libraries, namely those comprising the original UK Public Lending Right sample. They could fit data of three libraries by a simple geometric distribution; the negative binomial distribution provided reasonable approximations for four more of the libraries.

They went further and developed a model consisting of a mixture of negative binomial distributions. The idea that different frequency distributions might apply to different subsets of books within one library had already been advanced by Hindle and Worthington (1980). This yielded fits for seven other public libraries. Finally, they observed two 'mavericks', where data behaved so irregularly that they could not even be fitted by the mixed NBD model.

Finally, Burrell (1985a) applied the simple model to investigate the so-called *80/20 rule* (see also Subsection IV.7.1.1 for more details). He found the following relation :

$$y = x(1 + z\log x) \quad , \qquad [II.6.13]$$

where y is the proportion of circulations, x is the proportion of circulating items, ordered according to the decreasing frequency of circulation, and z is given by

$$z = \left(\mu_T \log\left(1 - \frac{1}{\mu_T}\right)\right)^{-1}, \ \mu_T > 1 \quad . \qquad [II.6.14]$$

Here μ_T is the mean number of circulations of circulating items over period T. These investigations concluded that over different lengths of time periods different proportions of circulating items will be required to achieve a given level of loans (say 80 %), and that conversely, a fixed proportion (say 50 %) of the circulating items will achieve different levels of loans. A similar investigation producing similar conclusions has been carried out by Egghe (1986b) but using a Lotka distribution to describe the frequency distribution of sources (circulating documents). In this case the relation between y and x is given by :

$$x = \frac{6}{\pi^2} e^{\gamma - \frac{\pi^2}{6} \mu_T (1-y)} , \qquad [II.6.15]$$

where $\gamma \approx 0.5772$ is Euler's number. For y = 0.8, this yields Table II.6.1.

Table II.6.1. Relation between the mean number of circulations and the proportion of circulating items needed to obtain 80 % of all circulations

μ_T	100 x% (Lotka distrib.)	100 x% (geometric distrib.)
1	77.9	-
2	56.1	56.8
3	40.4	52.1
4	29.0	49.8
5	20.9	48.6
6	15.0	47.7
7	10.8	47.1
8	7.8	46.7
9	5.6	46.4
10	4.0	46.1
∞	0.0	43.8

Table II.6.1 illustrates the fact that the higher μ_T is, the fewer sources one needs to obtain a fixed percentage of items. This implies, for instance, that in a public library one needs fewer books to obtain a fixed percentage of circulation than in, say, an academic library (since in public libraries the mean number of circulations is much higher than in academic libraries).

II.6.2.2. Incorporating ageing

Burrell (1985b) studied a modification of the mixed Poisson model. A drawback of the original formulation (as presented in the preceding subsection) is that it assumes that the lending rate of each item remains constant throughout time. In his modification Burrell further proposed that items become less desirable as time goes on, so that the lending process slows down. More precisely, he assumes that, for each item, desirability $\lambda(t)$ decays exponentially and that the rate of decay is the same for all items in the collection; hence $\lambda(t) = \lambda e^{-at}$; $t > 0$, $a > 0$. With these assumptions he is able to show that (Burrell (1985b; p.111-114)) :

(i) the mean number of loans in the n^{th} year is $\nu(\xi/a)(1-e^{-a})\ \theta^{n-1}$, with $\theta = e^{-a}$; from this formula we see that the annual mean number of loans decreases exponentially;

(ii) the frequency of circulation distribution over any given time is a negative binomial distribution;

(iii) the annual frequency of circulation distribution exhibits year-by-year change.

In particular, if the probability that an item will be on loan r times during the first n years is denoted by $P(X_n = r)$, he shows that

$$P(X_n = r) = \binom{r+\nu-1}{r} p_n^{\nu} q_n^{r} \ ; \quad r = 0,1,2,\ldots$$

where

$$p_n = (1 + \frac{\xi}{a}(1-\theta^n))^{-1} \ ,$$

and

$$q_n = 1 - p_n \ . \qquad [II.6.16]$$

Here ξ and ν are the parameters of the mixing gamma distribution and $\theta = e^{-a}$; $1 - \theta$ is the annual rate of decay and θ is said to be *the ageing factor*.

When only active items are considered and ageing effects are neglected

$$\lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} \beta(1+\xi n)^{-\nu} = 0 \ ,$$

according to Subsection II.6.2.1.

This means that without ageing factors every item will eventually be loaned out. On the other hand, if $\theta < 1$, i.e. if $a \neq 0$, and ageing is present, then

$$\lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} (1 + \frac{\xi}{a}(1-\theta^n))^{-\nu}$$

$$= (\frac{a}{a+\xi})^{\nu} \neq 0 \ . \qquad [II.6.17]$$

Consequently according to this model, if there is ageing, a certain proportion of the collection will never circulate. In practice, θ is estimated by the following fraction :

$$\frac{\text{total number of loans in year 2}}{\text{total number of loans in year 1}} ,$$

where we refer to a fixed collection.

This model was further used by Burrell (1987) to predict the future use of items in a collection and to investigate the consequences of different relegation procedures based on frequency-of-circulation data. The model itself obtained strong support from a citation study on obsolescence by Coughlin and Baran (1988). We also note the importance of relegation to effective browsing (Morse (1970)).

II.6.2.3. Morse's Markov model and the mixed Poisson model with ageing (Burrell (1986), Tague and Ajiferuke (1987))

In this subsection we will compare both models. Morse assumes that the number of uses of items in a fixed class of books follows a Poisson distribution, while the mixed Poisson model advocates the use of a negative binomial distribution. The crucial aspect of Morse's model is that year-by-year usage of items occurs according to a Markov chain. So in the Markov model the expected number of loans during year n+1 depends only on the number of loans in year n. On the other hand, according to the mixed Poisson model this expected number of loans depends on the total number of loans during the first n years. This implies that the latter model is more conservative.

Finally, the mean circulation N(m) during year n+1 of items circulating m times during year n is given by A + Bm. In Morse's original model (Morse (1968), Morse and Elston (1969)) A and B are assumed to be constant for a fixed class. Beheshti and Tague (1984) suggested an alternative expression for A, namely $A = A_0 + K_A n$ (hence depending on n). Kraft (1970) proposed yet another expression for A : $A = A_0 K_A^n$. Lastly, as a consequence of the mixed Poisson model with ageing (Burell (1986)) :

$$A = \frac{\nu\mu_n}{\nu + \mu_{n-1}}$$

and

$$B = A\nu^{-1} ,$$

where μ_n is the average number of loans per item in the n^{th} year and is given by $\mu_1\theta^{n-1}$ (parameters ν and θ have the same meaning as in the preceding subsection).

To test which of these dynamic models for library circulation fits real data best, Tague and Ajiferuke (1987) conducted an extensive investigation based on eleven years of circulation at the University of Saskatchewan. The results were rather disappointing as neither the Markov nor the mixed Poisson model fit the data. Discrepancies between models and data occurred largely in the class of non-circulating items.

We furthermore note that none of these models is fully explanatory since they assume certain distribution functions as building blocks for the models.

II.6.2.4. Further developments

Gelman and Sichel (1987) solved part of the problem that arose from the failure of the Tague-Ajiferuke study. They claim that no mixture of Poisson processes is appropriate for the modelling of book circulation data and show how an improved fit (in terms of χ^2-values) is achieved with a complicated model, namely a beta mixture of binomial distributions. Rather than going into detail (the functions are complicated and, moreover, unexplained), the interested reader is referred to the Gelman-Sichel paper (1987).

In a reaction Burrell (1990) admits the success of the beta-binomial model but observes that the model has three parameters, one of which, denoted S and interpreted as the maximum possible number of loans in a period, is difficult to estimate. Indeed, Gelman and Sichel (1987) used an ad hoc method, requiring visible inspection of the data. Therefore, it is not clear how reliable predictions can be made from this.

On the other hand, Burrell (1990) demonstrates (based on the Saskatchewan data) that, admitting the inadequacy of the mixed Poisson model with ageing, very accurate predictions could still be made. Therefore he concludes his paper by stating that the gamma-Poisson model can provide the library manager with useful advice in decision making : 'It may not be the correct model or even the best, but in general terms it works!'.

The idea of using various models for informetric processes in general was developed in Burrell (1988a). He suggested ways to use these models for predictive purposes.

The mathematically inclined reader will have enjoyed the evolution in library circulation models as described in the preceding sections. On the other hand, the practising librarian will probably point to Sandison's principles (1977) and ask 'Can't predictions be based on much simpler techniques?'. This was also recognised by Burrell (1988c), who developed a simple empirical Bayesian method which is indeed much easier to use. This approach was inspired by earlier work in informetrics by Brookes (1975), and is based on equations used in statistical and ecological studies by Goodman (1949) and Good and Toulmin (1956). For a complete account, the reader is advised to consult Burrell (1988c).

II.7. FUZZY SETS AND HEURISTIC METHODS IN LIBRARY MANAGEMENT

II.7.1. Fuzzy set theory

II.7.1.1. Imprecision in daily life

Planning often requires judgements on people or material to be used, time spent to reach subgoals and so on. Measurements that would be required to obtain objective estimates of probabilities are sometimes impossible or too costly in time and/or money. Subjective estimates are then necessary. Moreover, criteria are often only vaguely defined; for instance, what are 'popular old books'? Books that are at least 30 (20,40,100?) years old (written, published for the first time, for the last time?) and that circulate (where?) at least four (three?) times a year?

The first man who really saw and acted upon what has been called 'the fatal flaw of the denial of the existence of imprecision' was Lotfi A. Zadeh (1965) of the University of California, Berkeley. He is the founder and pioneer of that part of mathematics that is known as '*fuzzy set theory*'. Fuzzy set theory breaks through the two-valued classification of classical set theory (an item belongs or does not belong to a precisely defined set) and produces a measuring device to determine the degree to which an item belongs to a certain class. It is also able to cope with linguistic hedges such as very, rather, almost and slightly and with imprecise terms such as old, important, relevant and beautiful. A mathematical analysis based on fuzzy sets allows one to work with concepts that lie beyond the pale of classical mathematics, yielding finely nuanced results about the concepts being investigated.

II.7.1.2. Fuzzy set theory : basic concepts

Informally, a fuzzy set is a class in which the transition from membership to non-membership is gradual. A more precise definition of a fuzzy set is given as follows.

Let X be a set in the classical sense. Then a fuzzy subset of X is a set A of ordered pairs $(x,\mu_A(x)) \in X \times [0,1]$, where μ_A is a function $X \to [0,1]$ called the 'membership function' and $\mu_A(x)$ is the grade of membership of x in A. If A is a subset of X in the classical sense it can be considered as a fuzzy subset by taking μ_A equal to the characteristic function of A, which is 1 if $x \in A$ and 0 if $x \notin A$. Within fuzzy set theory subsets in the classical sense, i.e. those defined through a characteristic function, are termed 'crisp sets'.

An example. Let $X = \mathbb{R}^+$ (the positive real numbers). We may then define the fuzzy set B of small positive real numbers by :

$$\mu_B(x) \begin{cases} = (1 + (\frac{x}{5})^2)^{-1} , & x \leq 30 , \\ = 0 , & x > 30 . \end{cases}$$

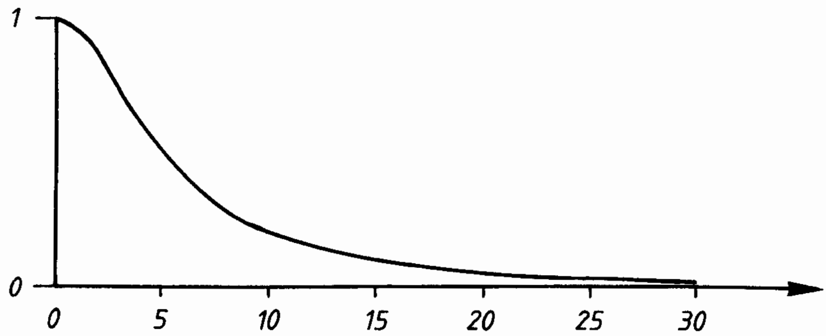

Fig.II.7.1 A fuzzy set of small positive real numbers

The fuzzy positive multiples of 5 can be defined as the fuzzy set $C \subset \mathbb{R}^+$, with

$$\mu_C = 1 - \frac{2}{5} (\min (x \pmod 5), 5 - (x \pmod 5)))$$

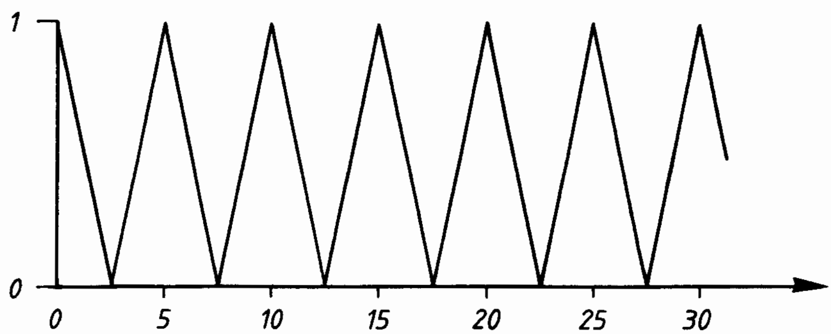

Fig.II.7.2 A fuzzy set of positive multiples of 5

The support of a fuzzy subset A of X is the crisp subset, denoted as Supp A and defined by Supp A = $\{x \in X;\ \mu_A(x) > 0\}$. In our examples Supp B = [0,30] and Supp C = $\mathbb{R}^+ \setminus \{5/2, 15/2, 25/2, \ldots\}$.

Two fuzzy subsets A_1 and A_2 of X are equal if $\mu_{A_1}(x) = \mu_{A_2}(x)$ for all $x \in X$. If A_1 and A_2 are fuzzy subsets, then A_1 is a subset of A_2, written $A_1 \sqsubset A_2$ if $\mu_{A_1}(x) \leq \mu_{A_2}(x)$ for all $x \in X$. The complement of a fuzzy subset A of X, denoted as A^c, is defined by

$$\mu_{A^c}(x) = 1 - \mu_A(x) , \qquad x \in X . \qquad [II.7.1]$$

The union of two fuzzy subsets, A_1 and A_2 of X, denoted as $A_1 \cup A_2$, is defined by

$$\mu_{A_1 \cup A_2}(x) = \max (\mu_{A_1}(x),\mu_{A_2}(x)) , \qquad x \in X . \qquad [II.7.2]$$

Finally, the intersection of two fuzzy subsets A_1 and A_2 of X, denoted as $A_1 \cap A_2$ is defined by

$$\mu_{A_1 \cap A_2}(x) = \min (\mu_{A_1}(x),\mu_{A_2}(x)) , \qquad x \in X . \qquad [II.7.3]$$

For example, the small multiples of five, shown by Fig.II.7.3 are the intersection of the small real numbers and the multiples of five.

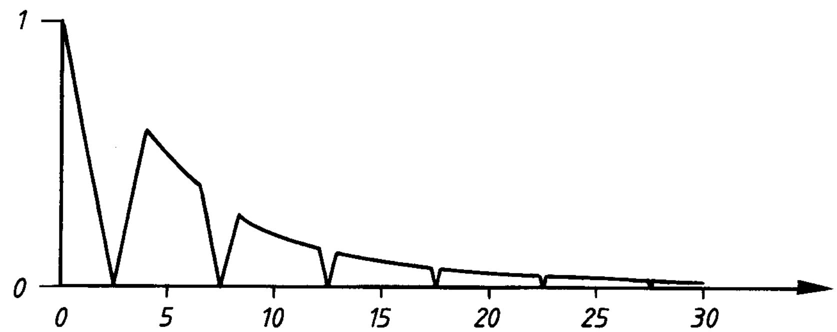

Fig.II.7.3 Small multiples of five

II.7.2. <u>A practical example : periodical binding decisions</u>

We will present an example, partly based on Robinson and Turner (1981) and Turner and O'Brien (1984) of the use of fuzzy set theory in practical library work. The problem we will consider is that of periodical binding decisions, which involve many vaguely defined variables. For example, a series of periodicals may have 'too many missing issues' or 'too low an impact on current research'. However, it is difficult if not impossible to come up with objective criteria for these variables.

A fuzzy-heuristic method, making explicit use of librarians' expertise, could be developed as follows. A small committee of experts is formed. For reasons of simplicity, we shall consider a team of two experts. Three criteria will be used :

- number of citations obtained by the journal, as measured by ISI's (Institute for Scientific Information) citation files;
- percentage of missing issues;
- number of circulations (local use) per issue.

Each committee member must decide on his/her membership function for each of these variables. So, although each of these criteria can be measured in an objective way, the interpretation of the measurements with respect to the ultimate binding decision is subjective and requires an application of concepts borrowed from fuzzy set theory. An example of membership functions of two experts is shown in Fig.II.7.4.

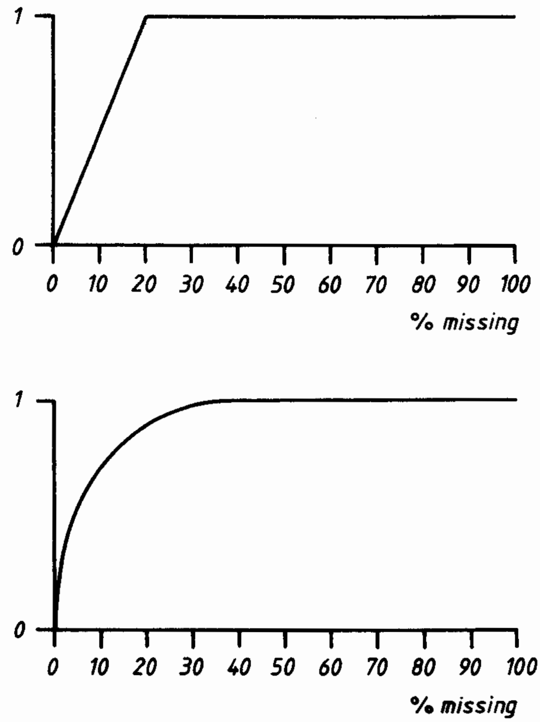

Fig.II.7.4 Membership functions of two experts for the variable 'too many missing issues'

When experts have decided on membership functions, every journal set can be judged on all criteria. This can now be done in a straightforward way and no longer requires a specific intellectual input.

Finally, each expert must also have decided, beforehand, on the relative importance of each of the three criteria, and the library committee must have decided on the relative importance of each expert (before the data were collected!). This leads to a ranking of journals according to their suitability for binding.

This approach produces a handy method for computerising an otherwise completely subjective decision procedure. Moreover, effective use is made of the knowledge and expertise of librarians who will not have the feeling that the computer has taken over their work. On the other hand, this procedure saves a lot of time since the librarian does not have to decide on every set of journals separately.

II.7.3. Notes and comments

Since 1976 (Tahani (1976)) people have tried to apply the concepts in fuzzy set theory to information retrieval. Although numerous theoretical papers and even reviews have appeared on this subject (see e.g. Buel (1982), Bookstein (1985), Rousseau (1985), Kerre et al. (1986)) very little use of these ideas has been made in actual implementations. Some people (Robertson (1978)) even gave well-founded criticisms of the application of fuzzy set concepts to information retrieval (IR), claiming that so-called fuzzy quantities proposed for use in IR should be quantified by probabilistic methods.

For a time, it seemed as if fuzzy set theory was a dead issue in IR. However, fuzzy set theory involves such attractive ideas that it came back in a different form. Today, its concepts and methods are being incorporated in relational databases that allow vague queries (Motro (1988)) and in knowledge-assisted document retrieval proposals (Biswas et al. (1987a, 1987b)). So, fuzzy sets are becoming a part of the artificial intelligence stream which is currently pervading every scientific and technological enterprise (Hofstadter (1980), Winston (1984)).